PMM U.S.S.R., Vol.47, No.4, pp.498-506, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 533.6.011:534

NON-LINEAR RESONANT OSCILLATIONS OF A GAS IN A TUBE UNDER THE ACTION OF A PERIODICALLY VARYING PRESSURE*

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One-dimensional non-linear motions of an ideal gas in a tube are considered. The tube is closed at one end, and the pressure, periodically varying with time, is specified at the other end. Non-linear asymptotic equations controlling the periodic oscillation of the gas are obtained for the frequency range close to the resonance frequencies. The complete problem of integrating the system of partial differential equations of gas dynamics is reduced to finding the solution of a single functional equation. Continuous solutions are constructed as well as solutions containing shock waves. It is shown that within the framework of the theory developed here, the solution of the problem in question is not unique for a fixed tube length and over a certain range of frequencies: two different solutions exist, one smooth, and another containing strong discontinuities.

A similar problem was studied earlier in /1—3/. The case of short tubes was analyzed most exhaustively in /1/ where the method of deformed Poincare-Lighthill coordinates was used to derive the non-linear gas oscillation equations (an analogous method of studying the resonant oscillations was given in /4/ for another class of problems). This method, however, does not enable discontinuous solutions to be constructed without additional assumptions. Unlike /4/, in the present case the problem of introducing the discontinuities is complicated even further by the fact that the shock waves generated within the flow vanish, after reflection from the boundary where the pressure is specified, in the form of a rarefaction wave. The centred rarefaction wave becomes a discontinuity as the boundary, and the gas pressure falls on passing through this discontinuity. This makes the procedure for constructing discontinuous solutions /1,2,4—11/ no longer suitable, since the stipulation that rarefaction discontinuities are forbidden is an essential factor when analyzing flows with shock waves used in the papers mentioned above. In addition to the above problems, problems of a fundamental nature also arise. The basic investigation on the non-linear oscillations of a gas, containing quantative results /1—11/, were carried out assuming the flow to be isentropic, and neglecting the changes in the Riemann invariant which occurs during the interaction between the characteristics and the shock waves. In the case of /3—11/ these assumptions represent an accurate result since the contribution of the above two effects to the asymptotic equation of gas oscillations are vanishingly small. In the present case the increments in the entropy and Riemann invariant in the shock waves have the same order of smallness as the basic terms retained in the equations of motion of the gas /1—3/. If this is indeed so, then the analysis of the oscillatory gas motions must change in a fundamental manner. The effect of the shock waves on the field of flow is also studied. It is shown that the increase in entropy at the discontinuities is not significant in the problem in question. The change in the value of the Riemann invariants, though causing the appearance of additional terms in the asymptotic equations of motion, has practically no effect on the final result. The solution obtained differs from one that disregards the change in the value of the Riemann invariant, by a higher-order infinitesimal. This is due to the fact that the contributions in question are different from zero only in narrow regions of the flow. It is established that the well-known area rule /3/ can be used to introduce the discontinuities with sufficient accuracy. All that has been said above makes it possible to use a correspondingly modified approach /3/ to solving the problem of resonant oscillations in a gas, caused by a periodically varying pressure.

1. Equations of motion.

The gas dynamic equations wirtten in characteristic form are /12/

$$\left(\frac{\partial u}{\partial t}\right)_\xi + \frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\xi = 0, \quad \left(\frac{\partial u}{\partial t}\right)_\eta - \frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\eta = 0, \quad \left(\frac{\partial s}{\partial t}\right)_\zeta = 0$$

where the differentiation operators acting along the characteristics are denoted by

*Prikl.Matem.Mekhan., Vol.47, No.4, pp.607-618, 1983

$$\left(\frac{\partial}{\partial t}\right)_\xi = \frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}, \quad \left(\frac{\partial}{\partial t}\right)_\eta = \frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x},$$
$$\left(\frac{\partial}{\partial t}\right)_\zeta = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

Here ξ, η, ζ are the characteristic variables for the families C^+, C^- and C^0, t is the time, x is a Cartesian coordinate, u is the velocity of the gas, p is the pressure, ρ is the density, s is the entropy and $a = (\partial p/\partial \rho)_s^{1/2}$ is the speed of sound. The parameters of the unperturbed gas at rest are denoted everywhere by a zero subscript.

Let us simplify the equations of motion by using the fact that the waves considered have small amplitude ε and the change of entropy within the field of flow is caused only by the appearance of weak shock waves and is therefore of the order of ε^3 /12/. Knowing the final result in advance,we retain in the equations of motion only the terms needed to compute the pressure with an accuracy of $O(\varepsilon^3)$. In this case it is obviously sufficient to determine the position of the characteristics with an accuracy of $O(\varepsilon^2)$, and terms of order ε^4 in the equations of motion can be neglected. The following relations obviously hold:

$$\rho = \rho(p, s_0) + O(\varepsilon^3), \ a = a(p, s_0) + O(\varepsilon^3)$$

This implies that in order to compute the pressure and velocity fields with the required accuracy, we can assume that ρ and a in the first two equations of motion are functions of the pressure when the value of the entropy is unperturbed, and we can reduce them to the standard form

$$\left(\frac{\partial J^+}{\partial t}\right)_\xi = 0, \quad \left(\frac{\partial J^-}{\partial t}\right)_\eta = 0; \quad J^\pm = u \pm \frac{2a}{\varkappa - 1}$$

where $J^\pm$ are the Riemann invariants and $\varkappa$ is the adiabatic exponent. We note again that now a is no longer the true speed of sound at the given point of the flow, and differs from it by a quantity of the order of $O(\varepsilon^3)$. Let us consider a boundary value problem suppose the condition of impermeability $u(X, t) = 0$, is given at the right end of the tube, and the pressure, periodically varying with time as

$$p(0, t) = p_0[1 + \delta f(t)], \ f(t + T) = f(t)$$

(δ is a small parameter, T is the period, and X denotes the length of the tube) at the left end. We introduce the dimensionless variables using the formulas

$$p = p_0(1 + \varepsilon p'), \ a = a_0(1 + \varepsilon a'), \ u = a_0 \varepsilon u'$$
$$J^\pm = a_0[\varepsilon J^{\pm\prime} \pm 2/(\varkappa - 1)], \ t = Tt', \ x = a_0 T x'$$

From now on we shall omit the primes accompanying the dimensionless variables. In the new variables the equations of motion and boundary conditions become

$$\left(\frac{\partial J^+}{\partial t}\right)_\xi = 0, \quad \left(\frac{\partial J^-}{\partial t}\right)_\eta = 0 \tag{1.1}$$
$$\left(\frac{\partial x}{\partial t}\right)_\xi = 1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi) + \frac{3-\varkappa}{4}\varepsilon J^-(\eta)$$
$$\left(\frac{\partial x}{\partial t}\right)_\eta = -1 + \frac{\varkappa+1}{4}\varepsilon J^-(\eta) + \frac{3-\varkappa}{4}\varepsilon J^+(\xi)$$
$$J^+(n, t) + J^-(n, t) = 0, \quad \frac{\varkappa\varepsilon}{2}[J^+(0, t) - J^-(0, t)] = \delta f(t) \tag{1.2}$$

We shall assume that the condition $4n = 2k + 1 + 4\Delta$, where k is an integer and $\Delta \ll 1$, holds for the dimensionless length of the tube $n = X/(a_0 T)$. This is the case of so-called quarter-wave resonance for which, as was established in /1–3/, the linear theory predicts an unbounded increase in the solution as $\Delta \to 0$.

The reason for the appearance of a resonance in the framework of the linear approximation can most simply be explained by considering Fig.1 ($\Delta = 0$). On the characteristics $C^\pm$ with slopes ± 1 we indicate the values of the invariants transported along them. By virtue of the boundary conditions we have

$$J_1^- = -J^+(\xi_0), \quad J_2^+ = -J^+(\xi_0) + 2\delta f(t_2)/(\varkappa\varepsilon), \quad J_3^- = -J^+(t_2),$$
$$J_4^+ = -J_2^+ + 2\delta f(t_4)/(\varkappa\varepsilon)$$

We see that when the characteristic C^+ traverses the tube twice, forwards and backwards, the invariant transferred by it acquires an increment

$$\Lambda = 2[f(\xi_0) - f(\xi_0 + k + {}^1/_2)]\,\delta/(\varkappa\varepsilon)$$

The periodicity of f implies that $2m$-tuple passage along the tube yields an increment in the value of the invariant of $m\Lambda$ and the latter quantity increases without limit as m increases. An exact analytic solution of the linear problem is given in /3/; in the case of $\Delta = 0$ the solution contains "secular" terms proportional to t. Actually, in the end there is no unlimited increase since, for long times, non-linear effects leading to stabilization of the system begin to exert their influence.

Below we shall consider short tubes ($n \sim 1$). We shall identify the characteristic variable ξ (η) with the instant when the corresponding characteristic C^+ (C^-) emerges from the left (right) boundary. Then integrating the equations for the characteristics we obtain

$$C^+: x = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t-\xi) + \frac{3-\varkappa}{4}\varepsilon I^+ \qquad (1.3)$$
$$C^-: x = n - \left[1 - \frac{\varkappa+1}{4}\varepsilon J^-(\eta)\right](t-\eta) + \frac{3-\varkappa}{4}\varepsilon I^-$$
$$I^+ = \int_\xi^t J^-(\eta)\,d\tau, \quad I^- = \int_\eta^t J^+(\xi)\,d\tau$$

The integral I^+ is evaluated at $\xi = \text{const}$, and I^- at $\eta = \text{const}$.

In /3–7, 9–12/ the interaction between the waves belonging to different families are neglected, since the quantities $I^\pm$ determining such interaction are of a higher order of smallness compared with the remaining terms in the equations for the characteristics. This substantially simplified the analysis of the almost resonant modes. It was shown in /1–3/ that this does not hold for the quarter-wave mode considered here.

We shall evaluate the integrals $I^\pm$ by successive approximations using the characteristics of the unperturbed gas $x = t - \xi$, $x = -t + \eta + n$ as the first approximation. Clearly, the integrals

$$I_0^+ = \int_\xi^t J^-(2\tau - \xi - n)\,d\tau, \quad I_0^- = \int_\eta^t J^+(2\tau - \eta - n)\,d\tau$$

on the piecewise smooth solutions differ from the exact values of $I^\pm$ by a quantity of order $O(\varepsilon)$. This implies that Eqs.(1.3), in which $I^\pm$ have been replaced by $I_0^\pm$ define in the (x, t)-plane curves which differ from the characteristics of the flow by quantities of order $O(\varepsilon^2)$. We will use the equations

$$C_1^+: x = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t-\xi) + \frac{3-\varkappa}{4}\varepsilon I_0^+ \qquad (1.4)$$
$$C_1^-: x = n - \left[1 - \frac{\varkappa+1}{4}\varepsilon J^-(\eta)\right](t-\eta) + \frac{3-\varkappa}{4}\varepsilon I_0^-$$

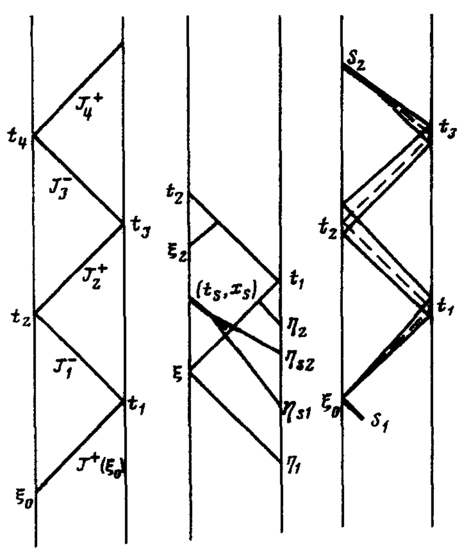

Fig.1 Fig.2 Fig.3

to compute the next approximation to $I^\pm$ using the notation of Fig.2.

Differentiating (1.4) for constant ξ and equating the right sides of the relations obtained, we establish a relation, accurate to $O(\varepsilon)$, connecting the time elapsed during the motion down the fixed characteristics C^+ with the increment in the value of the characteristic variable η of the waves belonging to the opposite family and intersecting the characteristic in question

$$dt = \frac{1}{2}\left\{d\eta - \frac{\varkappa+1}{8}\varepsilon J^+(\xi)\,d\eta - \frac{\varkappa+1}{8}\varepsilon J^-(\eta)\,d\eta + \frac{\varkappa+1}{4}(t-\eta)\,\varepsilon\, dJ^-(\eta)\right\} - \frac{3-\varkappa}{16}\varepsilon\left[J^+(\eta-n) + J^-(2t-\xi-n)\right]d\eta$$

The term in the square brackets can be neglected within the accuracy required. Indeed, in the regions where $\partial J^-/\partial\eta, \partial J^+/\partial\xi \sim 1$, it is of order ε by virtue of the first boundary condition (1.2). The extent of the zones of flow within which these derivatives are large is, as we shall see later, of order ε. Therefore, substituting the expression for dt into I^+ and integrating, we find that the contribution of the term in the square brackets to the interaction integral is of order ε^2.

When using the expression for dt obtained above, I^+ should be calculated with care, since the trajectory along which the integration is carried out may encounter on its way a shock wave as a result of intersecting the wave characteristics of the opposite family. Let the characteristic variables of the waves of the second family have the values η_{s2} and η_{s1} behind and in front of the front, at the point of contact (t_s, x_s) (Fig.2). Then

$$I_1^+ = \frac{1}{2}\Big\{I_1^-(\eta_1, \eta_{s1}) + I_1^-(\eta_{s2}, \eta_2) - \frac{\varkappa+1}{8}\varepsilon J^+(\xi) \times$$
$$[I_1^-(\eta_1, \eta_{s1}) + I_1^-(\eta_{s2}, \eta_2)] - \frac{\varkappa+1}{8}\varepsilon [I_2^-(\eta_1, \eta_{s1}) +$$
$$I_2^-(\eta_{s2}, \eta_2)] + \frac{\varkappa+1}{4}\varepsilon \Big(\int_{\eta_1}^{\eta_{s1}} + \int_{\eta_{s2}}^{\eta_2}\Big) J^-(\eta)(t-\eta)\, dJ^-(\eta)\Big\} =$$
$$\frac{1}{2}\Big\{I_1^-(\eta_1, \eta_2) - \frac{\varkappa+1}{8}\varepsilon J^+(\xi) I_1^-(\eta_1, \eta_2) -$$
$$\frac{\varkappa+1}{16}\varepsilon I_2^-(\eta_1, \eta_2) + \frac{\varkappa+1}{8}\varepsilon [J^{-2}(\eta_2)(t-\eta_2) - J^{-2}(\eta_1) \times$$
$$(\xi - \eta_1)]\Big\} - \frac{1}{2}\Big\{I_1^-(\eta_{s1}, \eta_{s2}) + \frac{\varkappa+1}{8}\varepsilon \times$$
$$[J^{-2}(\eta_{s2})(t_s - \eta_{s2}) - J^{-2}(\eta_{s1})(t_s - \eta_{s1})]\Big\}$$

Here and henceforth for brevity we will use the following notation:

$$I_1^+(\xi_1, \xi_2) = \int_{\xi_1}^{\xi_2} J^+(\xi)\, d\xi, \quad I_2^+(\xi_1, \xi_2) = \int_{\xi_1}^{\xi_2} J^{+2}(\xi)\, d\xi$$
$$I_1^-(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} J^-(\eta)\, d\eta, \quad I_2^-(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} J^{-2}(\eta)\, d\eta$$

By virtue of the obvious estimate $|\eta_{s1} - \eta_{s2}| = O(\varepsilon)$, the integrals in the formula for I_1^+ containing the small parameter ε as a factor are neglected in the interval (η_{s1}, η_{s2}). Analyzing the expressions in the second braces we find, that they are of order ε^2. Indeed, using the condition $x_s = x(\eta_{s1}) = x(\eta_{s2})$ and the second formula of (1.3) and recalling the estimate for $|\eta_{s1} - \eta_{s2}|$, we have

$$\Big[1 - \frac{\varkappa+1}{4}\varepsilon J^-(\eta_{s1})\Big](t_s - \eta_{s1}) = \Big[1 - \frac{\varkappa+1}{4}\varepsilon J^-(\eta_{s2})\Big](t_s - \eta_{s2}) + O(\varepsilon^2)$$

From this it follows that the expression considered is equal to

$$I_1^-(\eta_{s1}, \eta_{s2}) + \frac{\varkappa+1}{4}\varepsilon [J^-(\eta_{s1}) + J^-(\eta_{s2})](\eta_{s1} - \eta_{s2}) + O(\varepsilon^2) = O(\varepsilon^2)$$

by virtue of formula (1.9) of /3/ obtained for the relation at the discontinuity.

The second boundary condition of (1.2) yields $J^-(\eta_1) = J^+(\xi) + O(\delta/\varepsilon)$. Finally, we obtain the equation for the characteristic C^+ emerging from the left boundary at the instant ξ, written to within terms of order $O(\varepsilon^2)$ inclusive

$$C^+: x = \Big[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\Big](t-\xi) + \frac{3-\varkappa}{8}\varepsilon\Big\{I_1^-(\eta_1, \eta_2) -$$
$$\frac{\varkappa+1}{8}\varepsilon J^+(\xi) I_1^-(\eta_1, \eta_2) - \frac{\varkappa+1}{16}\varepsilon I_2^-(\eta_1, \eta_2) +$$
$$\frac{\varkappa+1}{8}\varepsilon [J^{-2}(\eta_2)(t-\eta_2) - J^{+2}(\xi) n]\Big\}$$

Putting $x = n$ we find the instant at which the characteristic in question reaches the right boundary

$$t_1 = \xi + n\Big[1 - \frac{\varkappa+1}{4}\varepsilon J^+(\xi) + \Big(\frac{\varkappa+1}{4}\varepsilon\Big)^2 J^{+2}(\xi)\Big] + \tag{1.5}$$
$$\frac{\varkappa-3}{8}\varepsilon\Big\{I_1^-(\eta_1, t_1) - \frac{\varkappa+1}{8}\varepsilon J^+(\xi) I_1^-(\eta_1, t_1) -$$
$$\frac{\varkappa+1}{16}\varepsilon I_2^-(\eta_1, t_1) - \frac{\varkappa+1}{8}\varepsilon J^{+2}(\xi) n\Big\}$$

Analogous arguments applied to the reflected characteristic C^- with certain alterations caused by the lack of symmetry in the boundary conditions, yield the equation

$$C^-: x = -\Big[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\Big](t - t_1) + \frac{3-\varkappa}{8}\Big\{I_1^+(\xi_1, \xi_2) - \tag{1.6}$$
$$\frac{\varkappa+1}{8}\varepsilon J^+(\xi) I_1^+(\xi, \xi_2) + \frac{13-3\varkappa}{16}\varepsilon I_2^+(\xi, \xi_2) -$$
$$\frac{\varkappa+1}{8}\varepsilon [J^{+2}(\xi_2)(t-\xi_2) - J^{+2}(\xi) n]\Big\} + n$$

The above relation yields the instant t_2 at which the characteristic C^+ reaches the left boundary after being reflected from the rigid wall. Substituting into (1.6) t_1 from (1.5) and putting $x = 0$, we obtain

$$2n = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t_2 - \xi) + \frac{3-\varkappa}{8}\varepsilon\Big\{I_1^-(\eta_1, t_1) - I_1^+(\xi, t_2) - \frac{\varkappa+1}{8}\varepsilon J^+(\xi)[I_1^-(\eta_1, t_1) - I_1^+(\xi, t_2)] - \frac{\varkappa+1}{4}\varepsilon J^{+2}(\xi)n - \frac{\varkappa+1}{16}\varepsilon I_2^-(\eta_1, t_1) - \frac{13-3\varkappa}{16}\varepsilon I_2^+(\xi, t_2)\Big\}$$

Transforming the last equation we obtain

$$t_2 = 2n\left[1 - \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right] + \varepsilon^2\alpha J^{+2}(\xi) + \varepsilon^2\beta + \xi \tag{1.7}$$

$$\alpha = \frac{\varkappa+1}{8}n\left(\varkappa + 1 + \frac{3-\varkappa}{4}\right), \quad \beta = \frac{(3-\varkappa)(13-3\varkappa)}{32}(2k+1)I_0$$

$$I_0 = \int_0^{1/2} J^{+2}(\xi)\,d\xi$$

which, together with the relation

$$J^+(t_2) = -J^+(\xi) + 2\delta f(t_2)/(\varkappa\varepsilon) \tag{1.8}$$

following from (1.2), forms a closed system of equations determining the solution of the problem of quarter-wave resonance.

To derive (1.7) from the previous equation we first transform the difference $I_1^-(\eta_1, t_1) - I_1^+(\xi, t_2)$. Here it is clear that in calculating the relations connecting the characteristic variables at the right and left boundary, to obtain the required accuracy it is sufficient to adopt the second equation of (1.4) for the characteristics C^-

$$\xi = n\left[1 + \frac{\varkappa+1}{4}\varepsilon J^-(\eta)\right] - \frac{3-\varkappa}{8}\varepsilon I_1^+(\eta - n, \xi) + \eta$$

Differentiating it and utilizing the chain of equations

$$J^+(\xi) = J^-(\eta) + \frac{2\delta}{\varkappa\varepsilon}f(\xi) = -J^+(\eta - n) + \frac{2\delta}{\varkappa\varepsilon}f(\xi) + O(\varepsilon)$$

which give the boundary conditions, we find

$$d\xi = d\eta + \frac{\varkappa+1}{4}\varepsilon n dJ^-(\eta) + \frac{3-\varkappa}{4}\varepsilon J^-(\eta)\,d\eta + O(\delta, \varepsilon^2)$$

Consequently

$$I_1^+(\xi, t_2) - I_1^-(\eta_1, t_1) = \frac{\varkappa+1}{8}\varepsilon[J^{-2}(t_1) - J^{-2}(\eta_1)] + \frac{3-\varkappa}{4}\varepsilon I_2^-(\eta_1, t_1) + O(\delta/\varepsilon) = \frac{3-\varkappa}{4}\varepsilon I_2^-(\eta_1, t_1) + O(\delta/\varepsilon, \varepsilon^2)$$

since the expression in the square brackets is of order $O(\delta/\varepsilon, \varepsilon)$ by virtue of the boundary conditions.

In deriving the last equation, for brevity we did not consider separately problem connected with the regions of large flow gradients and the formation of shock waves. The arguments used in deriving (1.5) are fully applicable here.

Consider the integrals $I_2^+(\xi, t_2)$, $I_2^-(\eta_1, t_1)$. From the boundary conditions it follows that in the regions of smoothness $J^+(\xi + k + 1/2) = -J^+(\xi)$, with an accuracy up to the terms of higher order of smallness. Since the solution is, by definition, a periodic function with period 1, $J^+(\xi + 1/2) = -J^+(\xi)$ and $J^{+2}(\xi + 1/2) = J^{+2}(\xi)$, i.e. the square of the solution is a periodic function with a period of ½. On the other hand, $t_2 = \xi + k + 1/2 + O(\Delta, \varepsilon)$, therefore we have, within the accuracy required,

$$I_2^+(\xi, t_2) = (2k+1)\int_0^{1/2} J^{+2}(\xi)\,d\xi = (2k+1)I_0 = \text{const}$$

Analogous arguments yield the same equation for $I_2^-(\eta_1, t_1)$. All this enables us to write

$$2n = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t_2 - \xi) - \frac{3-\varkappa}{8}\varepsilon^2 \times \left\{\frac{13-3\varkappa}{4}(2k+1)I_0 + \frac{\varkappa+1}{4}J^{+2}(\xi)n\right\}$$

from which we obtain (1.7).

2. Study of the oscillation equations.

In order to show more clearly the structure of the solution sought, we shall use the example given earlier in /3/. We write (1.7) and (1.8) for two instances t_2 and t_4, where t_2 is in the same relation to t_4, as ξ is to t_2

$$t_4 = 2n\left[1 - \frac{\varkappa+1}{4}\varepsilon J^+(t_2)\right] + \varepsilon^2\alpha J^{+2}(t_2) + \varepsilon^2\beta + t_2 \tag{2.1}$$
$$J^+(t_4) = -J^+(t_2) + 2\delta f(t_4)/(\varkappa\varepsilon)$$

Let us combine the first equation of (2.1) with (1.7) and subtract (1.8) from the second equation of (2.1). This yields

$$t_4 = 2n\left[2 - \frac{\varkappa+1}{2\varkappa}\delta f(t_2)\right] + 2\varepsilon^2\alpha J^{+2}(\xi) + 2\varepsilon^2\beta + \xi \tag{2.2}$$
$$J^+(t_4) = J^+(\xi) + 2\delta\,[f(t_4) - f(t_2)]/(\varkappa\varepsilon)$$

The above formulas were derived with help of the boundary conditions (1.2), and terms of higher order of smallness were neglected. The operation corresponds to Fig.1, with the linear equations of the characteristics replaced by their non-linear counterparts. The extraordinary term (the second term in square brackets) is retained in the formula for t_4 for clarity. The usefulness of the transformation carried out becomes clear when we note that t_4 differs from ξ by an integral number of periods plus a small increment. This enables us, by virtue of the periodicity of the solution sought, to expand $J^+(t_4)$ in the smooth regions in series. Subsitituting the first two terms of this expansion into (2.2) and putting $\delta = \varepsilon^3$, $\Delta = \sigma\varepsilon^2$, $\sigma \lesssim 1$, we obtain, with the accuracy required,

$$J^{+\prime}(\xi)\ [4\sigma + 2\alpha J^{+2}(\xi) + 2\beta] = 2\,[f(\xi) - f(\xi + {}^1/_2)]/\varkappa \tag{2.3}$$

The conclusion that $\delta \sim \varepsilon^3$, agrees with the results obtained in /1—3/.

In /3/, where quarter-wave resonance in long tubes was studied, a mistake was made when deriving (3.11), which is similar to (2.3) of the present paper. The quantity $f(t_4)$ was replaced there by $f(t_2 - {}^1/_2)$, which is incorrect, since in this case we have $\varepsilon n \sim 1$. Nevertheless, this has no effect on the results obtained in /3/.

Below, we shall assume that $f(\xi) = -2\varkappa \sin 2\pi\xi/(\varkappa + 1)$. The integration of (2.3) yields a third-degree algebraic equation in the unknown function $J^+(\xi)$

$$J^{+3}(\xi) + 3pJ^+(\xi) - 2q\cos 2\pi\xi + C = 0 \tag{2.4}$$
$$p = (2\sigma + \beta)/\alpha,\ q = 3/[(\varkappa + 1)\pi\alpha]$$

where C is an arbitrary constant. Its solution is a one-parameter family of curves, from which we can construct, in principle, knowing the value of the constant C at every segment of smoothness, a continuous or discontinuous solution of the problem in question. Here the problem arises of the choice of C and of the method of introducing strong discontinuities into the solution.

In /4/ where half-wave resonance was studied, the problem was successfully overcome, and one of the decisive aspects enabling discontinuous solutions to be uniquely constructed was found to be the physical requirement forbidding discontinuities across which a fall in pressure occurs. Moreover, the discontinuity appearing in the field of flow continued its periodic motion along the tube and did not disappear, and this also simplified the analysis of the flow. The opposite situation arises in the problem of quarter-wave resonance. From (1.2) we see that on reaching the left boundary the discontinuity vanishes, being reflected locally in the form of a centred rarefaction wave. The latter represents a discontinuity at the boundary, and the passage through this discontinuity is accompanied by a fall in pressure. We find that in describing the oscillations in terms of the differential equations we encounter, in addition to the discontinuities described above, discontinuities interchanging between the regions with large gradients in which the gas is compressed. Such regions appear when a bundle of characteristics generated by a shock wave after it has traversed the tube completely in both directions impinges on the boundary. All this makes not only makes the approach used in /4/ unsuitable in the present case, but also the stricter methods developed later in /1,5—11/. Undoubtedly the method described in /13/ offers one of the possible ways of solving the problem in question, although it requires substantial reformulation before it can be used in the present case.

Below we shall follow the method used in /3/, which represents essentially the method of characteristics, the position of which is determined not using the difference equations, but directly at finite distances from their origins. Modifications to the algorithm given in /3/ are required due to the need to take into account the interaction between the waves of different families. Equations (1.7) and (1.8), although incorporating such interaction, were obtained ignoring the variation in the Riemann invariants at the discontinuities. We shall discuss this problem and derive a rule governing the introduction of strong discontinuities into the solution.

Let the characteristic C^+ intersect the shock wave propagating in the backward direction. As a result of the interaction that occurs, the value of the invariant carried along it acquires an increment $\varepsilon^3\Delta J^+$, $\Delta J^+ \sim 1$ [12]. This change in the value affects the position of the characteristic (1.7) by a shift of the same order ε^3, which can be neglected. Regarding the contribution $\varepsilon^3\Delta J^+$ to (1.8), it appears to be significant by virtue of (2.3), since it is of the same order of smallness as the terms retained in (2.4).

The assessment of the influence of the shock waves on the pressure and velocity fields requires information on the width of the bundle of characteristics interacting with the shock wave. We shall use the following argument. The shock wave S_1 (Fig.3) is reflected from the left boundary in the form of a packet of rarefaction waves which, dispersing during its motion along the tube, is reflected from the rigid wall and returns to the left boundary in the form of a zone of width $h = O(\varepsilon)$. Here the derivatives $\partial J^+/\partial\xi$ are of the order of $1/\varepsilon$. The zone and its inverse image on the right boundary represent the regions of large gradients discussed in Sect.1. The packet propagates from the left boundary in the form of a compression wave, the characteristics of which converge and intersect, after the second reflection from the right boundary, at a distance l from the left boundary, generating the shock wave S_2.

We will estimate l using (1.5)–(1.7). In Fig.3 the dashed lines show the separate characteristic of the bundle. After repeated reflection from the rigid wall its equation, using the notation of Fig.3, becomes

$$t = \xi_0 + 4n - x\left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi_0)\right] - \frac{3-\varkappa}{8}\varepsilon\{I_1^-(t_1, t_3) - I_1^+(t_2, \xi_2)\} + O(\varepsilon^2) \tag{2.5}$$

where ξ_2 is the characteristic variable corresponding to the actual instant of time t, and the quantity $J^+(\xi_0)$ varies from J_1^+ to J_2^+, representing the values of the invariants transported by the outer characteristics of the bundle. The instant of a discontinuity is generated corresponds to the condition of "overturning" the wave profile $\partial t/\partial J^+(\xi_0) = 0$. Differentiating (2.5) we obtain the estimate $l = O(\varepsilon)$. It is clear from Fig.3 that the extension l_1 of the bundle of characteristics interacting with the shock wave is of the same order.

The simplest way to assess the effect of a change undergone by the Riemann invariant in the shock wave on the solution of the problem in question is to turn to the differential equation (2.3). This certainly leads to the appearance of an additional term ΔJ^+ on the right side of (2.3), which will differ from zero, as was shown above, in the narrow extension zones $\sim\varepsilon$, lying near the shock wave and the region of compression where the gradients are large. It is clear therefore that inclusion of ΔJ^+ will introduce corrections into the solution, which are of the same order of smallness ε and can therefore be neglected.

Let us derive the rule for introducing strong discontinuities into the solution. From (2.5) we see that up to the instant the shock wave appears, the characteristics of the initial compression zone are grouped in the region of extension $O(\varepsilon^2)$. Consequently their further relative displacement, dependent on the interaction integrals and the quadratic terms, will be of order $O(\varepsilon^3)$. Therefore, from the first appearance of the discontinuity up to its arrival at the left boundary, the motion of the characteristics, and hence the discontinuities, are described with the required accuracy of the order of $O(\varepsilon^2)$ by the formulas for simple waves. This in turn dictates the rule for introducing a shock wave as a simple wave into the region where the solution is multivalued. The position of the region is determined by the condition for the areas of the figures bounded by the curve $J^+(x, t)$ for fixed x and by the shock wave, and lying on opposite sides of the latter, to be equal.

Let us derive a formula analogous to (1.9) /3/ for the case in question. If the shock wave reaches the left boundary at the instant t_s and the characteristics intersecting it have initial coordinates ξ_{s1}, ξ_{s2}, then the area rule state that

$$\int_{J_{s1}^+}^{J_{s2}^+} (t - t_s)\, dJ^+ = 0, \quad J_{s1}^+ = J^+(\xi_{s1}), \quad J_{s2}^+ = J^+(\xi_{s2})$$

Then by virtue of (1.7) the following series of equations holds:

$$\int_{J_{s1}^+}^{J_{s2}^+} (t - t_s)\, dJ^+ = \frac{\varkappa+1}{4} n\varepsilon (J_{s2}^{+2} - J_{s1}^{+2}) - \frac{2\alpha\varepsilon^2}{3}(J_{s2}^{+3} - J_{s1}^{+3}) - I_1^+(\xi_{s1}, \xi_{s2}) = 0$$

Using this formula we can show that the integral law of conservation of the invariant holds for the discontinuous solutions

$$\int_{t_1}^{t_2} J^-(0, t)\, dt = -\int_{\xi_1}^{\xi_2} J^+(\xi)\, d\xi + \frac{\varkappa+1}{4} n\varepsilon [J^{+2}(\xi_2) - J^{+2}(\xi_1)] - \frac{2\alpha\varepsilon^2}{3}[J^{+3}(\xi_2) - J^{+3}(\xi_1)] \tag{2.6}$$

(t_1 and t_2 are obtained from(1.7) where ξ is replaced by ξ_1 and ξ_2 respectively). In the case of smooth solutions, (2.6) becomes a trivial corollary of (1.7).

Below we shall need some information on the roots of the cubic equation (2.4) with $C = 0$.

In the limit as $\varepsilon \to 0, \sigma = \text{const}$, the quantity q tends to a constant value and p depends not only on the length of the tube determined by σ, but also on the solution itslef through β. The roots of (2.4) are found using the Cardano formulas.

The graphs shown in Fig.4 and 5 by dashed lines represent schematically the behaviour of the solutions. If $p > 0$, then a unique real smooth solution exists of the type shown in Fig.4c. The amplitude of the oscillations (by which we shall mean the highest numerical value of J^+) increases as p increases. On passing to $p < 0$ we have a single continuous multivalued solution (Fig.4a and b), and this holds as long as $p^3 + q^2 > 0$. As p decreases the points at which the solution turns $(\partial J^+/\partial \xi = \infty)$ approach each other, and merge when $p^3 + q^2 = 0$. Further reduction in p leads to the solution separating into three smooth branches (Fig.5a and b) and the amplitude of the branch passing through zero decreases as p decreases, while the amplitudes of the remaining two branches increase. The value $p = -1.53$ at $k = 0$ corresponds to the limit mode $p^3 + q^2 = 0$ Figs.4a and 5b depict the situations close to the limit when the approach is made from different directions, namely from $p^3 + q^2 > 0$ and $p^3 + q^2 < 0$, respectively.

3. Numerical construction of the solution. Periodic solutions of (1.7) and (1.8) were constructed using the scheme given in /3/. First a certain distribution $J^+(\xi)$, was defined on a segment of unit length $[\xi_0, \xi_0 + 1]$ satisfying the conditions

$$\int_{\xi_0}^{\xi_0+1} J^+(\xi) = 0, \quad J^+(\xi_0) = J^+(\xi_0 + 1) \tag{3.1}$$

By the transformation (1.7) the segment $[\xi_0, \xi_0 + 1]$ becomes $[t(\xi_0), t(\xi_0 + 1)]$, which is obviously also of unit length. If the solution was "tilted" during this process, shock waves were introduced in the regions of multivaluedness according to the rule given in Sect.2. The quantities $J^+(\xi)$ in the section $[t(\xi_0), t(\xi_0 + 1)]$ were then found from (1.8). Clearly, the new function constructed on $[t(\xi_0), t(\xi_0 + 1)]$ also satisfied conditions (3.1) and the first equation follows from the law of conservation of the invariant (1.6), while the second one is trivial. In computing, the first condition of (3.1) holds only approximately and has been used in the computation as a control. New values of $J^+(\xi)$ obtained were continued periodically onto $[\xi_0, \xi_0 + 1]$, and the procedure described above was repeated until the determination was completed.

The method was used to carry out the computations for various values of ε and tube lengths n. As a result it was established that for every fixed ε a range of values of n exists close to the resonance values where two essentially different oscillatory modes are realized. In one case the solution contains strong discontinuities, in the other case it is continuous, and the amplitude of the discontinuous solution exceeds that of the continuous solution. In Figs. 4 and 5 the solid lines represent the "oscillograms"

$$J^\circ(\tau) = {}^1/_4\,(\varkappa + 1)\, J^+(2\pi t) \text{ for } \delta = 10^{-3},\ k = 0,\ \varkappa = 1.4$$

The dashed lines depict the solutions (reduced to the same variables J° and τ). The constants p and q were computed from the corresponding numerical solutions. The graphs indicated in Figs.4 and 5 by the same letters correspond to single values of σ, namely to -0.38 for 4a and 5a, and -0.195 for 4b and 5b. The values of p for Fig.4a and b are -1.44 and -0.47, for Fig.5a and b they are -3.6 and -1.67, and for Fig.4c they are $-0.002, \sigma = 0.097$. When $\sigma = -0.385$, was used in the computations, solutions with discontinuities could not be constructed and a continuous mode was developed in the computations.

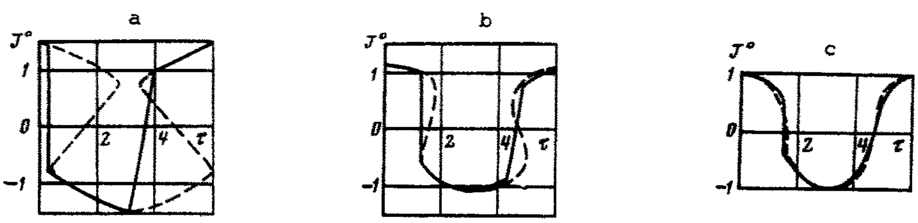

Fig.4

We can conclude that two modes exist within the accuracy requires up to the instant when a limit situation $p^3 + q^2 = 0$ occurs. We note that in accordance with (2.4) the critical value $p = -1.53$, while the computations gave $p = -1.44$. The computations carried out for other values of ε confirm the correctness of the deductions made. In Fig.6 we show a graph illustrating the dependence of the amplitude A on σ. We see that two oscillation modes exist in the range $-0.38 \leqslant \sigma \leqslant -0.195$.

The situation described above bears a qualitative resemblance to the case of forced osicllations of a pendulum with damping /14/. We note that the above discussion can be applied to the case considered in /1/, with the effect of the reflection coefficient taken into account.

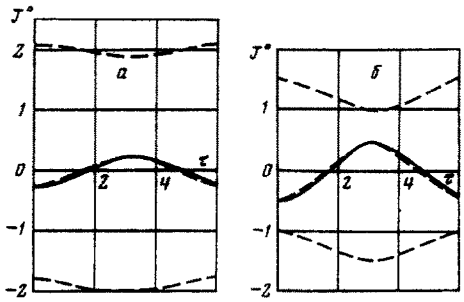

fig.5

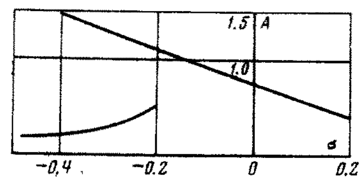

Fig.6

The author thanks A.N. Kraiko for his interest and V.E. Fortov for his supervision.

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Translated by L.K.